Pergamon
www.elsevier.com/locate/jappmathmech

PII: S0021-8928(01)00018-1

J. Appl. Maths Mechs, Vol. 65, No. 1, pp. 163–166, 2001

0021-8928/01/$—see front matter

THE DETERMINATION OF THE PERIODIC MOTIONS OF SYSTEMS WITH DELAY†

L. Z. FISHMAN

Nizhnii Novgorod

(*Received* 24 *March* 1999)

Continuing the development of results previously obtained for systems with delay described by first-order differential equations with delay [1], a system without delay is constructed which enables the periodic motions of systems with delay to be found.

Problems concerned with determining the periodic motions generated from equilibrium states of systems with delay have been considered before [2–4]. For systems with delay there are no reasonably simple and convenient ways of finding periodic motions as there are for systems without delay. It is therefore essential to establish new ways of finding periodic motions of systems with delay, described even by a first-order system with delay.

Let us consider a system with delay described by a first-order differential equation with delay

$$\dot{x} = a(\varepsilon)x + b(\varepsilon)x(t-\tau) + F(\varepsilon, x, x(t-\tau)) \tag{1}$$

where x is a scalar, ε is a parameter, and the dot stands for differentiation with respect to t.

Let us assume that $F(0, 0, 0) = 0$ and that the analytical function $F(\varepsilon, x_1, x_2)$ may be expanded in the neighbourhood of the point $x_1 = x_2 = 0$ in a series beginning with terms of at least the second order in (x_1, x_2), of the following form

$$\begin{gathered} F(\varepsilon, x_1, x_2) = F^{(2)} + F^{(3)} + \ldots \\ F^{(2)}(\varepsilon, x_1, x_2) = a_{ik}(\varepsilon)x_i x_k, \quad F^{(3)}(\varepsilon, x_1, x_2) = a_{ikp}(\varepsilon)x_i x_k x_p \end{gathered} \tag{2}$$

where a_{ik} and a_{ikp} are coefficients that depend on the parameter ε. Throughout this paper, repeated subscripts i, k (i, k, p) indicate summation over all $1 \leqslant i \leqslant k \leqslant 2$ $(1 \leqslant i \leqslant k \leqslant p \leqslant 2)$.

Suppose the characteristic equation

$$\Delta(p) = p - a(\varepsilon) - b(\varepsilon)\exp(-p\tau) = 0$$

when $\varepsilon = 0$ has roots $p_{1,2} = \pm i\omega$ and p_j which satisfy the condition $\operatorname{Re} p_j(0) < -\sigma < 0$.

Assume that

$$d\operatorname{Re} p_{1,2}(\varepsilon)/d\varepsilon|_{\varepsilon=0} > 0$$

In what follows we will construct, for systems with delay described by Eq. (1), a second-order, system without delay, using which one can approximately find limit cycles of Eq. (1), depending on the values of ε. The limit cycles of Eq. (1), determined for small ε, may then be extended to finite values of ε by numerical continuation with respect to ε.

We write Eq. (1) in operator form [5]

$$dx_t(\theta)/dt = Ax_t(\theta) + R(\varepsilon, x_t(\theta)), \quad x_t(\theta) = x(t+\theta) \tag{3}$$

$$Ax_t(\theta) = \begin{cases} dx_t(\theta)/d\theta, & -\tau \leqslant \theta < 0 \\ a(\varepsilon)x_t(0) + b(\varepsilon)x_t(-\tau), & \theta = 0 \end{cases}$$

$$R(\varepsilon, x_t(\theta)) = \begin{cases} 0, & -\tau \leqslant \theta < 0 \\ F(\varepsilon, x_t(0), x_t(-\tau)), & \theta = 0 \end{cases}$$

where $x(t)$ is a solution of Eq. (1) for $t > 0$ with continuously differentiable initial function $x_0(\theta) = \varphi(\theta)$.

Consider the functionals

†*Prikl. Mat. Mekh.* Vol. 65, No. 1, pp. 165–168, 2001.

$$y_j(t,\varepsilon)=f_j(x_t(\theta),\varepsilon)=x_t(0)-b(\varepsilon)\int_0^{-\tau}\exp(-p_j\tau)x_t(\nu)d\nu$$

and the functions

$$b_j(\theta,\varepsilon)=\exp(p_j(\varepsilon)\theta)/\Delta_j \qquad (\Delta_j=1+b(\alpha)\tau p_j(\varepsilon))$$

Proceeding as in the previous paper [5], using the change of variables

$$y_j(t)=f_j(x_t(\theta),\varepsilon), \qquad z_t(\theta)=x_t(\theta)-b_l(\theta,\varepsilon)y_l(t)$$

(throughout, a repeated subscripts l indicates summation from 1 to 2), we replace Eqs (3) by the system of equations

$$\dot y_j=p_jy_j+H_{11}, \qquad dz_t(\theta)/dt=Az_t(\theta)+H_{12} \tag{4}$$

where

$$H_{1n}=F(\varepsilon,b_l(0,\varepsilon)y_l+z_t(0),\ b_l(-\tau,\varepsilon)y_l+z_t(-\tau))\{\delta_{1n}+\delta_{2n}[(\delta(\theta)-(b_1(\theta,\varepsilon)+(b_2(\theta,\varepsilon)]\}$$

$\delta(\theta)=0$ for $-\tau\leqslant\theta<0$, $\delta(0)=1$ and δ_{mn} is the Kronecker delta.

Suppose $F^{(2)}(\varepsilon,x_1,x_2)=0$. In that case, the truncated system without delay is obtained [5] by putting $z_t(0)=z_t(-\tau)=0$ in the first two equations of system (4), and has the form

$$\dot y_j=p_j(\varepsilon)y_j+F^{(3)}(\varepsilon,\alpha_{11}y_1+\alpha_{12}y_2,\alpha_{2l}y_l+\alpha_{22}y_2), \quad j=1,2 \tag{5}$$

$$\alpha_{j1}=b_j(0,\varepsilon), \quad \alpha_{j2}=b_j(-\tau,\varepsilon).$$

Taking the last relation of (2) into consideration, we obtain from system (5)

$$\begin{aligned}\dot y_j&=p_j(\varepsilon)y_j+a_{ikp}(\varepsilon)[(\alpha_{il}y_l)(\alpha_{kl}y_l)(\alpha_{pl}y_l)]=\\&=p_j(\varepsilon)y_j+D_{30}(\varepsilon)y_1^3+D_{21}(\varepsilon)y_1^2y_2+D_{12}(\varepsilon)y_1y_2^2+D_{03}(\varepsilon)y_2^3\end{aligned} \tag{6}$$

where

$$\begin{aligned}D_{30}(\varepsilon)&=a_{ikp}(\varepsilon)\alpha_{i1}\alpha_{k1}\alpha_{p1}\\D_{21}(\varepsilon)&=a_{ikp}(\varepsilon)(\alpha_{i1}\alpha_{k1}\alpha_{p2}+\alpha_{i1}\alpha_{k2}\alpha_{p1}+\alpha_{i2}\alpha_{k1}\alpha_{p1})\\D_{12}(\varepsilon)&=a_{ikp}(\varepsilon)(\alpha_{i2}\alpha_{k2}\alpha_{p1}+\alpha_{i1}\alpha_{k2}\alpha_{p2}+\alpha_{i2}\alpha_{k1}\alpha_{p2})\\D_{03}(\varepsilon)&=a_{ikp}(\varepsilon)\alpha_{i2}\alpha_{k2}\alpha_{p2}\end{aligned} \tag{7}$$

System (5) is a system of differential equations which, up to terms $y_1^ry_2^q$ of order $r+q=3$, is identical with a system on a two-dimensional stable central manifold, which exists in the neighbourhood of the equilibrium state $x=0$ of Eq. (1) for sufficiently small ε [3, 4], and which we will denote by σ.

If when $\varepsilon=0$ the quantity g_1 defined for Eq. (1), which is similar to the first Lyapunov number, does not vanish, then as ε is increased from $\varepsilon<0$ to $\varepsilon>0$, a limit cycle $\Gamma(\varepsilon)$ lying on σ is generated from the equilibrium state $x=0$, or contracts to that state. Since g_1 for $\varepsilon=0$ is also simultaneously the first Lyapunov number of system (6), it follows that, as ε is increased from $\varepsilon<0$ to $\varepsilon>0$, a limit cycle $\Gamma_0(\varepsilon)$ is generated from the equilibrium state $y_1=y_2=0$ or contracts to that state. For sufficiently small ε, the limit cycles $\Gamma_0(\varepsilon)$ and $\Gamma(\varepsilon)$ are close together. Let (y_1^0,y_2^0) be a point of $\Gamma_0(\varepsilon)$. Then for small ε the function

$$x_0(t)=b_l(\theta,\varepsilon)y_l^0, \quad (\theta\in[-\tau,0])$$

may be taken as an approximate initial function for the limit cycle $\Gamma(\varepsilon)$. As ε increases from $\varepsilon=0$ or as ε decreases from $\varepsilon=0$, the limit cycle $\Gamma(\varepsilon)$ of Eq. (1) may be determined approximately, e.g. by numerical continuation with respect to the parameter ε.

Now consider the case $F^{(2)}\neq0$. Following earlier arguments [5], we introduce the following change of variables in system (4)

$$z_t(\theta)=v_t(\theta)+\gamma(\theta,y_1,y_2,\varepsilon) \quad \gamma=\sum_{r+q=2}d_{rq}(\theta,\varepsilon)y_1^ry_2^q \tag{8}$$

(the values of $d_{rq}(\theta,\varepsilon)$ will be found below).

In the new variables, system (4) becomes

$$\dot{y}_j = p_j y_j + F(\varepsilon, \alpha_{l1}y_1 + \gamma(0, y_1, y_2, \varepsilon) + v_t(0),\ \alpha_{l2}y_1 + \gamma(-\tau, y_1, y_2, \varepsilon) + v_t(-\tau))$$
$$dv_t(\theta)/dt = A\,v_t(\theta) + H_2(y_1, y_2, v_t(\theta), \varepsilon) \tag{9}$$
$$H_2 = A\gamma + H_1 - (\partial\gamma/\partial y_l)\dot{y}_l, \qquad H_1 = B_{20}(\theta,\varepsilon)y_1^2 + B_{11}(\theta,\varepsilon)y_1y_2 + B_{02}(\theta,\varepsilon)y_2^2 + \dots$$

where the dots stand for terms $y_1^r y_2^q$ with $r + q \geqslant 3$. The coefficients $B_{rq}(\theta, \varepsilon)$, where $r + q = 2$, have the form

$$B_{rq}(\theta,\varepsilon) = A_{rq}(\varepsilon)(\delta(\theta) - (b_1(\theta,\varepsilon) + b_2(\theta,\varepsilon)))$$

where $A_{rq}(\varepsilon)$ are the coefficients of the quadratic terms $y_1^r y_2^q$ $(r + q = 2)$ in the function $F(\varepsilon, \alpha_{l1}y_1, \alpha_{l2}y_1)$.

Equating the coefficients of the quadratic terms $y_1^r y_2^q$ in the function H_2 to zero [5], we obtain equations for $d_{rq}(\theta, \varepsilon)$ $(r + q = 2)$:

$$[J\lambda_{rq} - A]d_{rq}(\theta) = B_{rq}(\theta,\varepsilon); \quad \lambda_{20} = 2p_1(\varepsilon), \quad \lambda_{11} = p_1(\varepsilon) + p_2(\varepsilon), \quad \lambda_{02} = 2p_2(\varepsilon) \tag{10}$$

where J is the identity operator.

Equation (10) with $r + q = 2$ yields

$$d_{rq}(0) = A_{rq}(1 - \alpha_{11} - \alpha_{12} + b(\varepsilon)L_{rq})/\Delta(\lambda_{rq}), \qquad d_{rq}(-\tau) = \exp(-\lambda_{rq}\tau)d_{rq}(0) + A_{rq}L_{rq} \tag{11}$$
$$L_{rq} = \sum_{j=1}^{2} \frac{\exp(-p_j\tau) - \exp(-\lambda_{rq}\tau)}{(p_j - \lambda_{rq})\Delta_j}, \qquad \Delta(\lambda_{rq}) = \lambda_{rq} - a(\varepsilon) - b(\varepsilon)\exp(-\lambda_{rq}\tau)$$

The truncated second-order system without delay is obtained in this case from system (9) when $v_t(0) = v_t(-\tau) = 0$

$$\dot{y}_j = p_j(\varepsilon)y_j + Q(\varepsilon, y_1, y_2), \quad j = 1, 2 \tag{12}$$

where

$$Q(\varepsilon, y_1, y_2) = F(\varepsilon, \psi_1, \psi_2) = a_{ik}(\varepsilon)\psi_i\psi_k + a_{ikp}(\varepsilon)\psi_i\psi_k\psi_p \tag{13}$$
$$\psi_j = \alpha_{j1}y_1 + \alpha_{j2}y_2 + \sum_{r+q=2} d_{rq}^{(j)}y_1^r y_2^q, \qquad d_{rq}^{(1)} = d_{rq}(0,\varepsilon), \quad d_{rq}^{(2)} = d_{rq}(-\tau,\varepsilon), \quad r+q=2, \quad j=1,2$$

The function $Q(\varepsilon, y_1, y_2)$ in (12) has the form

$$Q(\varepsilon, y_1, y_2) = \sum_{2\leqslant k} \sum_{r+q=k} A_{rq}(\varepsilon)y_1^r y_2^q \tag{14}$$

From (13), multiplying ψ_i, ψ_k and ψ_p, we obtain

$$A_{20}(\varepsilon) = a_{ik}(\varepsilon)\alpha_{i1}\alpha_{k1}, \quad A_{11}(\varepsilon) = a_{ik}(\varepsilon)(\alpha_{i1}\alpha_{k2} + \alpha_{k1}\alpha_{i2}) \tag{15}$$
$$A_{02}(\varepsilon) = a_{ik}(\varepsilon)\alpha_{i2}\alpha_{k2}, \quad A_{rq}(\varepsilon) = D_{rq}(\varepsilon) + C_{rq}(\varepsilon) \quad (r+q=3)$$

The functions $D_{rq}(\varepsilon)$ are given by formulae (7), and

$$C_{30}(\varepsilon) = a_{ik}(\varepsilon)(\alpha_{i1}d_{20}^{(k)} + \alpha_{k1}d_{20}^{(i)})$$
$$C_{21}(\varepsilon) = a_{ik}(\varepsilon)(\alpha_{i1}d_{11}^{(k)} + \alpha_{k1}d_{11}^{(i)} + \alpha_{i2}d_{20}^{(k)} + \alpha_{k2}d_{20}^{(i)})$$
$$C_{12}(\varepsilon) = a_{ik}(\varepsilon)(\alpha_{i2}d_{11}^{(k)} + \alpha_{k2}d_{11}^{(i)} + \alpha_{i1}d_{02}^{(k)} + \alpha_{k1}d_{02}^{(i)})$$
$$C_{03}(\varepsilon) = a_{ik}(\varepsilon)(\alpha_{i2}d_{02}^{(k)} + \alpha_{k2}d_{02}^{(i)})$$

Consider the system

$$\dot{y}_j = p_j(\varepsilon)y_j + Q^*(\varepsilon, y_1, y_2), \quad j = 1, 2 \tag{16}$$
$$Q^* = \sum_{2\leqslant k\leqslant 3} \sum_{r+q=k} A_{rq}(\varepsilon)y_1^r y_2^q.$$

As in the case $F_2 = 0$, system (16) with $\varepsilon = 0$ has a first Lyapunov number g_1, which is identical with a quantity similar to the first Lyapunov number for Eq. (1). Hence it follows that in the present case one obtains results, relating to the determination of limit cycles $\Gamma(\varepsilon)$ of Eq. (1) with the help of system (16), similar to those obtained above for the case $F_2 = 0$.

As an example, consider the equation

$$\dot{x} = -\varepsilon \sin(x(t-1)) \tag{17}$$

The characteristic quasi-polynomial

$$p + \varepsilon e^{-p} = 0 \tag{18}$$

with $\varepsilon = \pi/2$ has two roots $p_{1,2} = \pm i\pi/2$ and the other roots have Re $p_j < -\sigma < 0$. When $\varepsilon = \pi/2$, $g_1 < 0$ for Eq. (17). Hence it follows that when ε is increased from $\varepsilon = \pi/2$, a stable limit cycle is generated from the equilibrium state $x = 0$ of Eq. (17). It has been found by the technique described above for $\varepsilon = 1.9$, 2.5 and 3. In all these cases one obtains stable periodic solutions in the (x, t) plane, similar to sinusoids with period 4 and amplitudes 0.2, 1.7 and 2.3. These periodic solutions are stable limit cycles of Eq. (17).

This research was supported financially by the Russian Foundation for Basic Research (98-01-00635).

REFERENCES

1. FISHMAN, L. Z., Criteria of dangerous and safe boundaries of the stability domain of equations with delay. *Prikl. Mat. Mekh.*, 1997, **61**, 4, 706–713.
2. CHAFEE, N., A bifurcation problem for a functional differential equation of finitely retarded type. *J. Math. Anal. Appl.*, 1971, **35**, 26, 312–348.
3. HALE J., *Theory of Functional Differential Equations*. Springer, New York, 1977.
4. HASSARD, B. D., KAZARINOFF, N. D. and WAN, J. -H., *Theory and Applications of Hopf Bifurcation*. Cambridge University Press, London, 1981.
5. SHIMANOV, S. N., The critical case of a pair of pure imaginary roots for systems with aftereffect. *Sibirsk. Mat. Zh.*, 1961, **2**, 3, 467–480.
6. EL'SGOL'TS, L. E. and NORKIN, S. B., *Introduction to the Theory of Differential Equations with Retarded Argument*. Nauka, Moscow, 1971.

Translated by D.L.